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On adiabatic N-soliton interactions and trace identities

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Abstract. We compare the Hamiltonian properties of the N-soliton solutions of the NLSE in the adiabatic approximation and show how it matches the Hamiltonian formulation for the complex Toda chain which describes the adiabatic N-soliton interactions.

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1 Introduction

It is well known (see [1–4] and the references therein) that the N-soliton train interactions in the adiabatic approximation is described by the complex Toda chain (CTC). More precisely this means the following. Let us consider the solution of the nonlinear Schrödinger equation (NLSE):

$$\mathrm{i}u_t + \frac{1}{2}u_{xx} + |u|^2 u(x,t) = 0, \tag{1}$$

satisfying the initial condition

$$u(x, t=0) = \sum_{k=1}^{N} u_k^{1\mathrm{s}}(x, t=0), \tag{2}$$

where $u_k^{1\mathrm{s}}(x,t)$ is the one-soliton solution of the NLSE:

$$\begin{aligned} &u_k^{1\mathrm{s}}(x,t) = 2\nu_k \mathrm{e}^{\mathrm{i}\tilde{\phi}_k} \operatorname{sech} z_k, \qquad z_k = 2\nu_k(x - \mu_k t - \xi_{0,k}),\\ &\tilde{\phi}_k = \frac{\mu_k}{\nu_k} z_k + \delta_k(t), \qquad \delta_k(t) = 2(\mu_k^2 + \nu_k^2)t + \delta_{0,k}, \end{aligned} \tag{3}$$

and ν_k, μ_k, $\xi_{0,k}$ and $\delta_{0,k}$ are the kth soliton amplitude, velocity, initial position and phase. The solution determined by (2) is known as the N-soliton train (NST). It is an approximation to the exact N-soliton solution to the NLSE because some (small) part of its energy comes from excitations over the continuous spectrum [5].

The adiabatic approximation means that the solitons are well separated and can be viewed as separate entities. It is valid provided the soliton parameters: ν_k, μ_k, ξ_k and δ_k, satisfy [6]:

$$\begin{aligned} |\nu_k - \nu_0| \ll \nu_0, \qquad |\mu_k - \mu_0| \ll \mu_0,\\ \nu_0 r_0 \gg 1, \qquad |\nu_k - \nu_0| r_0 \ll 1, \end{aligned} \tag{4}$$

where $\nu_0 = \sum_{k=1}^{N} \nu_k/N$ and $\mu_0 = \sum_{k=1}^{N} \mu_k/N$ are the average amplitude and velocity and r_0 is the average distance between the neighbouring solitons. We also assume that initially the soliton are quasi-equidistant, *i.e.*:

$$\xi_{0,k+1} - \xi_{0,k} \simeq r_0, \qquad k = 1, \ldots, N-1. \tag{5}$$

This approximation allows one to view the evolution of the NST as a slow evolution of its $4N$ parameters. The main result in [1–4] consists in proving that if we define $q_k(t)$ by:

$$\begin{aligned} q_{k+1}(t) - q_k(t) = -2\nu_0(\xi_{k+1} - \xi_k) + \ln 4\nu_0^2\\ + \mathrm{i}[\pi + 2\mu_0(\xi_{k+1} - \xi_k) - \delta_{k+1} + \delta_k], \end{aligned} \tag{6}$$

then the evolution of the NST is described by:

$$\frac{\mathrm{d}^2 q_k}{\mathrm{d}\tau^2} = \mathrm{e}^{q_{k+1} - q_k} - \mathrm{e}^{q_k - q_{k-1}}, \qquad k = 1, \ldots, N, \tag{7}$$

assuming $\tau = 4\nu_0 t$ and $\mathrm{e}^{q_{N+1}} \equiv \mathrm{e}^{q_0} \equiv 0$. The system (7) known as the CTC generalizes the well known Toda chain, see [7].

The NST is close to the exact N-soliton solution which is a completely integrable Hamiltonian system with $2N$ degrees of freedom. We shall see that the same is true also for the CTC system. Our aim in the present paper is to make a comparative analysis between these two Hamiltonian systems.

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2 Preliminaries

The NLSE (1) is one of the famous equations integrated by the inverse scattering method, see [8,9]. The Lax operator here is given by the Zakharov-Shabat system:

$$L(\lambda)\psi \equiv \mathrm{i}\frac{\mathrm{d}\psi}{\mathrm{d}x} + (q(x,t) - \lambda\sigma_3)\,\psi(x,t,\lambda) = 0, \quad (8)$$

$$q(x,t) = \begin{pmatrix} 0 & u^*(x,t) \\ u(x,t) & 0 \end{pmatrix}, \qquad \lim_{x\to\pm\infty} q(x,t) = 0.$$

Indeed the NLSE (1) can be written down in the Lax form, *i.e.* it can be viewed as the compatibility condition

$$[L(\lambda), M(\lambda)] = 0, \quad (9)$$

where $M(\lambda)$ determines the time evolution of the family of common eigenfunctions $\psi(x,t,\lambda)$. Equation (9) may be used to express the coefficients of $M(\lambda)$ in terms of q, q_x and q_{xx} thus verifying that (9) and (1) are equivalent.

One of the consequences of the Lax representation is that each solution of the NLSE can be parametrized through the spectral data of (8). The Lax operator $L(\lambda)$ (8) has a continuous spectrum which fills up the real axis in the complex λ-plane. It may also have a finite number of pairs of discrete eigenvalues $\lambda_k^+ = (\lambda_k^-)^*$, $\operatorname{Im}\lambda_k^+ > 0$, $j = 1,\ldots,N$ which correspond to the N-soliton solution of (1). The NST is generically characterized by N pairs of discrete eigenvalues and some small contribution from the continuous spectrum which will be neglected.

Let us denote the scattering matrix $T(\lambda)$ of (8) by

$$T(\lambda,t) = \begin{pmatrix} a^+(\lambda) & -b^-(\lambda,t) \\ b^+(\lambda,t) & a^-(\lambda) \end{pmatrix}. \quad (10)$$

and assume that $u(x,t)$ satisfies the NLSE. Then the elements of $T(\lambda,t)$ satisfy linear evolution equations [9]:

$$\mathrm{i}\frac{\mathrm{d}a^\pm}{\mathrm{d}t} = 0, \qquad \mathrm{i}\frac{\mathrm{d}b^\pm}{\mathrm{d}t} \mp 2\lambda^2 b^\pm(\lambda,t) = 0. \quad (11)$$

An important tool to study the interrelations between the solutions of the NLSE and the scattering data of $L(\lambda)$ are the trace identities, see [9]. They relate the two possible ways to write down integrals of motion: a) in terms of the solution of NLSE and b) in terms of the scattering data. Below we will need the first three of them:

$$C_1 = \int_{-\infty}^{\infty} \mathrm{d}x|u|^2, \qquad C_2 = \frac{\mathrm{i}}{2}\int_{-\infty}^{\infty} \mathrm{d}x(u_x^* u - u^* u_x),$$

$$C_3 = \int_{-\infty}^{\infty} \mathrm{d}x(|u_x|^2 - |u|^4), \quad \ldots, \quad (12)$$

$$C_k = \frac{2^k \mathrm{i}}{k}\sum_{j=1}^N \left((\lambda_j^-)^k - (\lambda_j^+)^k\right), \qquad k = 1,\ldots,N, \quad (13)$$

where $\lambda_j^\pm$ are the discrete eigenvalues of $L(\lambda)$.

The action-angle variables for the NLSE related to the discrete spectrum of L are given by (see [8,9]):

$$\eta_k^\pm = \pm\lambda_k^\pm, \qquad \kappa_k^\pm = \mp\ln b_k^\pm, \qquad k = 1,\ldots,N, \quad (14)$$

where $\lambda_k^\pm$ are the discrete eigenvalues of (8) and $b_k^\pm$ are the normalization coefficients of the corresponding Jost solutions. Their evolution is provided by ($\tau = 4\nu_0 t$):

$$\frac{\mathrm{d}\eta_k^\pm}{\mathrm{d}\tau} = 0, \qquad \frac{\mathrm{d}\kappa_k^\pm}{\mathrm{d}\tau} = \mp\frac{(\lambda_k^\pm)^2}{2\mathrm{i}\nu_0}\,. \quad (15)$$

The Hamiltonian equals:

$$H_{\mathrm{NLSE}} = \frac{1}{4}C_3 = \frac{2\mathrm{i}}{3}\sum_{k=1}^N \left((\lambda_k^-)^3 - (\lambda_k^+)^3\right). \quad (16)$$

Of course these results are strictly valid for the exact N-soliton solutions.

Next we need the Hamiltonian formulation of the TC. The Hamiltonian is provided by:

$$H_{\mathrm{TC}} = \sum_{k=1}^N \frac{p_k^2}{2} + \sum_{k=1}^{N-1} \mathrm{e}^{q_{k+1}-q_k}, \quad (17)$$

where p_k and q_k are canonically conjugate dynamical variables:

$$\{p_k, q_s\} = \delta_{ks}. \quad (18)$$

The Lax representation of TC is in the form, see [7]:

$$\frac{\mathrm{d}L_{\mathrm{TC}}}{\mathrm{d}\tau} = [L_{\mathrm{TC}}, M], \quad (19)$$

$$L_{\mathrm{TC}} = \sum_{k=1}^N B_k E_{kk} + \sum_{k=1}^{N-1} A_k(E_{k,k+1} + E_{k-1,k}), \quad (20)$$

$$B_k = \frac{p_k}{2}, \qquad A_k = \frac{1}{2}\mathrm{e}^{(q_{k+1}-q_k)/2}, \quad (21)$$

where the $N\times N$ matrices $(E_{kj})_{mn} = \delta_{km}\delta_{jn}$.

The TC is a completely integrable Hamiltonian system whose action-angle variables are determined by [7]

$$\{\zeta_k, \qquad \rho_k = \ln r_k, \qquad k = 1,\ldots,N\}, \quad (22)$$

where ζ_k are the eigenvalues of the Lax matrix (19) and $r_k = v_1^{(k)}$ is the first component of the corresponding normalized eigenvector $\boldsymbol{v}^{(k)}$, *i.e.*:

$$L_{\mathrm{TC}}\boldsymbol{v}^{(k)} = \zeta_k \boldsymbol{v}^{(k)}, \qquad (\boldsymbol{v}^{(k)}, \boldsymbol{v}^{(k)}) = 1. \quad (23)$$

If A_k and q_k are real, then both ζ_k and ρ_k also take real values; in addition ζ_k are pair-wize distinct [7]. The evolution of the action-angle variables of the TC is given by:

$$\frac{\mathrm{d}\zeta_k}{\mathrm{d}\tau} = 0, \qquad \frac{\mathrm{d}\rho_k}{\mathrm{d}\tau} = -\zeta_k. \quad (24)$$

The trace identities for the TC system follow from the relations:

$$I_p = \mathrm{tr}\,(L_{\mathrm{TC}})^p = \sum_{k=1}^N (\zeta_k)^p, \qquad p = 1,\ldots,N. \quad (25)$$

In particular,

$$I_2 = \sum_{k=1}^{N} B_k^2 + \sum_{k=1}^{N-1} 2A_k^2 = \sum_{k=1}^{N} \zeta_k^2, \qquad (26)$$

and in view of (21) and (17) we get $I_2 = (1/2)H_{\mathrm{TC}}$.

3 Hamiltonian formulation of CTC

The step that brings us from TC to CTC is called complexification, see [10]. This is a rather formal step after which the dynamical variables, and also the Hamiltonian become complex. The next step which is possible due to the fact that H_{TC} depends analytically on the dynamical variables is the following. We consider the complexified phase space $\mathcal{M}_{\mathbb{C}}$ with $2N$ complex dimensions as a real $4N$-dimensional phase space spanned by the real and imaginary parts of $p_k = p_{0,k} + \mathrm{i}p_{1,k}$ and $q_k = q_{0,k} + \mathrm{i}q_{1,k}$. The symplectic structure in this $4N$-dimensional real phase space is introduced by:

$$\{p_{0,k}, q_{0,s}\} = \delta_{ks}, \qquad \{p_{1,k}, q_{1,s}\} = -\delta_{ks}. \qquad (27)$$

Then the Hamiltonian equations of motion provided by (27) and the Hamiltonian:

$$H_{\mathrm{CTC}} = \mathrm{Re}\left(\sum_{k=1}^{N} \frac{p_k^2}{2} + \sum_{k=1}^{N-1} \mathrm{e}^{q_{k+1}-q_k}\right) = \sum_{k=1}^{N} \frac{p_{0,k}^2 - p_{1,k}^2}{2} + \sum_{k=1}^{N-1} \mathrm{e}^{q_{0,k+1}-q_{0,k}} \cos(q_{1,k+1} - q_{1,k}), \qquad (28)$$

coincide with the equations of motion for the CTC. Thus we can view the CTC as a standard Hamiltonian system with $2N$ degrees of freedom, see, *e.g.* [4].

Due to the analyticity of H_{TC} a number of properties of the Toda chain easily generalize to the CTC. These include the integrability properties, the action-angle variables, the explicit form of the solutions and the trace identities (25). The difference is that now the integrals of motion I_p as well as the eigenvalues ζ_k become complex valued. In particular $I_1 = N(\mu_0 + \mathrm{i}\nu_0)/2$ is the simplest integral of motion for CTC and I_2 provides the Hamiltonian H_{CTC}.

It is natural to expect that the integrals of motion C_k of the NLSE in the adiabatic approximation will go into integrals of motion of the CTC, up to terms whose t-derivatives are of the order of $\epsilon^{3/2}\ln\epsilon$. To check this we will evaluate C_k, $k = 1, 2, 3$ in terms of the NST parameters. To this end we will evaluate the integrals in (12) with $u(x, 0)$ given by (2). This gives us three types of terms: i) 'local' in k terms that will be of order of 1; they give the contribution from u_k only; ii) terms coming from the overlap of two neighbouring solitons; they depend on k and $k+1$ and have orders $\epsilon\ln\epsilon$ and ϵ. These we will evaluate explicitly. iii) various other terms of higher orders in ϵ which will be dropped. Skipping the details (see the Appendix) we give the answer:

$$C_1 = \sum_{k=1}^{N} 4\nu_k - 8\sum_{k=1}^{N-1} \mathrm{Re}\, R_{k,k+1}, \qquad (29)$$

$$C_2 = \sum_{k=1}^{N} 8\nu_k\mu_k - 16\sum_{k=1}^{N-1} \mathrm{Re}\,(\lambda_0 R_{k,k+1}) - 8\sum_{k=1}^{N-1} \mathrm{Im}\,\mathrm{e}^{q_{k+1}-q_k}, \qquad (30)$$

$$C_3 = \sum_{k=1}^{N} 16\nu_k\left(\mu_k^2 - \frac{\nu_k^2}{3}\right) - 32\sum_{k=1}^{N-1} \mathrm{Re}\,(\lambda_0^2 R_{k,k+1}) - 32\sum_{k=1}^{N-1} \mathrm{Im}\,(\lambda_0 \mathrm{e}^{q_{k+1}-q_k}), \qquad (31)$$

where $\lambda_0 = \mu_0 + \mathrm{i}\nu_0$, $R_{k,k+1} = (\xi_{k+1} - \xi_k)\mathrm{e}^{q_{k+1}-q_k}$, and $q_{k+1} - q_k$ is given in (6).

Our first result is the explicit form of the terms $R_{k,k+1}$ which are of the order $\epsilon\ln\epsilon$; they are known as 'secular' terms [4]. Next we note that from (29–31) there follows the relation:

$$4(\mu_0^2 + \nu_0^2)C_1 - 4\mu_0 C_2 + C_3 = \mathrm{const.} + 32\nu_0 H_{\mathrm{CTC}}, \quad (32)$$

where the constant term in the right hand side is an expression depending only on μ_0 and ν_0. Indeed, it is easy to see that in the left hand side of (32) the secular terms cancel out.

One can expect that all the higher integrals I_p can be obtained as the adiabatic approximations from a conveniently chosen linear combination of C_k's.

4 Evolutions of the action-angle variables

The adiabatic approximation imposes restrictions on soliton parameters of the NST (6) which reflect on the scattering data of the Zakharov-Shabat system (10). Skipping the details of their derivation we just formulate them:

$$|\lambda_k^+ - \lambda_0|^2 \simeq \mathcal{O}(\epsilon), \qquad \lambda_0 = \frac{1}{N}\sum_{k=1}^{N} \lambda_k^+. \qquad (33)$$

and $\lambda_k^- = (\lambda_k^+)^*$. In other words the eigenvalues of $L(\lambda)$ in the $\mathbb{C}_+$ are clustered around their average value; the radius of the region is determined by $\epsilon^{1/2}$. The conditions on the angle variables are similar:

$$|\kappa_k^+ - \kappa_0| \simeq \mathcal{O}(\epsilon), \qquad \kappa_0 = \frac{1}{N}\sum_{k=1}^{N} \kappa_k^+. \qquad (34)$$

The same type of constraints hold also for the scattering data of L_{CTC} (22):

$$|\zeta_k^+ - \zeta_0|^2 \simeq \mathcal{O}(\epsilon), \qquad |\rho_k^+ - \rho_0| \simeq \mathcal{O}(\epsilon), \qquad (35)$$

where $\zeta_0 = \frac{1}{N}\sum_{k=1}^{N} \kappa_k^+$ and $\rho_0 = \frac{1}{N}\sum_{k=1}^{N} \rho_k^+$.

However the evolution of the action-angle variables for the NLSE (15) and for the CTC (24) look substantially different. This is due to the fact that the dispersion laws for the NLSE and CTC are given by:

$$f_{\mathrm{NLSE}}(\lambda) = \lambda^2, \qquad f_{\mathrm{CTC}}(\lambda) = \lambda, \tag{36}$$

respectively. In order to make the comparison we need to separate the dynamical variables for the 'center of mass' and introduce the sets $\{\lambda_0, \tilde{\lambda}_k^+, \kappa_0, \tilde{\kappa}_0^+\}$ and $\{\zeta_0, \tilde{\zeta}_k^+, \rho_0, \tilde{\rho}_0^+\}$ where

$$\tilde{\lambda}_k^+ = \lambda_k^+ - \lambda_0, \qquad \tilde{\kappa}_k^+ = \kappa_k^+ - \kappa_0, \tag{37}$$

$$\tilde{\zeta}_k^+ = \zeta_k^+ - \zeta_0, \qquad \tilde{\rho}_k^+ = \rho_k^+ - \rho_0. \tag{38}$$

It is easy to check that ρ_0 and $\tilde{\rho}_k$ satisfy:

$$\frac{\mathrm{d}\rho_0}{\mathrm{d}\tau} = \zeta_0, \qquad \frac{\mathrm{d}\tilde{\rho}_k}{\mathrm{d}\tau} = -\tilde{\zeta}_k. \tag{39}$$

Analogous simple calculation shows that the evolution of κ_0 and $\tilde{\kappa}_k$ is given by:

$$\begin{aligned}\frac{\mathrm{d}\kappa_0}{\mathrm{d}\tau} &= -\frac{1}{2\mathrm{i}\nu_0}\left(\lambda_0^2 - \langle\tilde{\lambda}^2\rangle\right),\\ \frac{\mathrm{d}\tilde{\kappa}_k}{\mathrm{d}\tau} &= -\frac{\lambda_0}{\mathrm{i}\nu_0}\tilde{\lambda}_k - \frac{1}{2\mathrm{i}\nu_0}\left(\tilde{\lambda}_k^2 - \langle\tilde{\lambda}^2\rangle\right).\end{aligned} \tag{40}$$

where $\langle\tilde{\lambda}^2\rangle = 1/N\sum_{k=1}^N \tilde{\lambda}_k^2 = \mathcal{O}(\epsilon)$. Note that the leading terms in the right hand sides of (40) correspond to a linear dispersion law characteristic for the CTC. So if we prove that $\zeta_k \simeq \lambda_k^+$ then we prove also the equivalence of the NST dynamics to the CTC one as completely integrable Hamiltonian systems. It is now easy to see the relation between (39) and (40) or (42). Indeed, let us fix up the reference frame of the NST in such a way that $\mu_0 = \mathrm{Re}\,\lambda_0 = 0$. Then $\lambda_0 = \mathrm{i}\nu_0$ and neglecting the terms of order $\mathcal{O}(\epsilon)$ in the right hand side of (40) we recover (39).

Let us now briefly discuss the NST dynamics for the higher NLSE. They are characterized by dispersion law $F(\lambda)$ which is cubic or higher order polynomial in λ. Then the angle variables evolve according to:

$$\frac{\mathrm{d}\kappa_k^+}{\mathrm{d}\tau} = -\frac{F(\lambda_k^+)}{2\mathrm{i}\nu_0}, \tag{41}$$

which means that

$$\begin{aligned}\frac{\mathrm{d}\kappa_0}{\mathrm{d}t} &= -\frac{F(\lambda_0)}{2\mathrm{i}\nu_0} - \frac{F''(\lambda_0)}{4\mathrm{i}\nu_0}\langle\tilde{\lambda}^2\rangle + \ldots,\\ \frac{\mathrm{d}\tilde{\kappa}_k}{\mathrm{d}t} &= -\frac{F'(\lambda_0)}{2\mathrm{i}\nu_0}\tilde{\lambda}_k^+ - \frac{F''(\lambda_0)}{4\mathrm{i}\nu_0}\left(\tilde{\lambda}_k^2 - \langle\tilde{\lambda}^2\rangle\right) + \ldots,\end{aligned} \tag{42}$$

where the dots mean terms of higher order in ϵ. The equivalence between (39) and (42) is established in analogy with the previous case. The only difference is in the coefficient in front of $\tilde{\lambda}_k^+$ which can be taken care of.

From this point of view one can understand the universality of the CTC in the sense that it describes the NST dynamics for the higher NLSE. Recently this understanding was extended by showing that the CTC is responsible for the interactions of the NST related to the modified NLSE [11] and to the multicomponent NLSE [12].

5 Stability of the N-soliton bound states

In this section we briefly analyze some aspects of the stability of NST bound states. In [1–3] we noted that the asymptotic regime of the NST is determined by ζ_k. In particular, if $\mathrm{Re}\,\zeta_k = 0$ for all $k = 1, \ldots, N$ then all N solitons form a bound state (BS); if all $\mathrm{Re}\,\zeta_k$ are pair-wize distinct then we have an asymptotically free (AF) regime in which each soliton moves uniformly with its asymptotic velocity $\mu_{k,\mathrm{as}} = 2\mathrm{Re}\,\zeta_k$.

Let us now remember that the Hamiltonian of CTC can be expressed in terms of ζ_k as follows:

$$\begin{aligned}(1/2)H_{\mathrm{CTC}} &= \mathrm{Re}\,\mathrm{tr}\,(L_{\mathrm{CTC}})^2 = \sum_{k=1}^N\left((\mathrm{Re}\,\zeta_k)^2 - (\mathrm{Im}\,\zeta_k)^2\right)\\ &= -N\,(\mathrm{Im}\,(\zeta_0))^2 + \sum_{k=1}^N\left((\mathrm{Re}\,\tilde{\zeta}_k)^2 - (\mathrm{Im}\,\tilde{\zeta}_k)^2\right)\end{aligned} \tag{43}$$

In deriving the last expression we used the assumption that the average velocity $\mathrm{Re}\,\zeta_0 = 0$.

Let us parameterize the phase space of the NST by the action-angle variables of the CTC and let us consider in it the sphere determined by

$$\mathcal{S} \equiv \sum_{k=1}^N |\zeta_k|^2 = \sum_{k=1}^N\left(\mathrm{Re}\,(\zeta_k)^2 + \mathrm{Im}\,(\zeta_k)^2\right) = \mathrm{const.} \tag{44}$$

Note that $\mathcal{S}$ is an integral surface different from the one of the Hamiltonian H_{CTC}. On $\mathcal{S}$ the N-soliton bound state will have minimal energy which could explain its stability. In all other regimes: AF or mixed regimes one or more of the eigenvalues ζ_k will have non-vanishing real parts and therefore their energy will be greater than the one for the BS regime.

This stability shows also by the fact that for some choices of the parameters all N solitons may travel quasi-equidistantly for times much longer than the critical time $T_{\mathrm{cr}} \simeq \epsilon^{-1}\ln\epsilon$, for more details see [13]. We illustrate this fact in Figure 1 in which $r_0 = 8$ and $T_{\mathrm{cr}} \simeq 372$. We have checked that the same type of behaviour persists to distances of 12 000 dispersion lengths.

Such quasi-equidistant propagation, if it can be achieved experimentally, might be important for soliton-based fiber optics communications. The problem is that the configurations of the soliton parameters responsible for it are rather difficult to achieve.

Appendix

In this appendix we will illustrate how it is possible to evaluate explicitly one of the typical integrals

$$J_{kn} = \int_{-\infty}^{\infty}\frac{\mathrm{d}x\,\mathrm{e}^{\mathrm{i}(\tilde{\phi}_k - \tilde{\phi}_n)}}{\cosh z_k \cosh z_n}, \tag{45}$$

that appear in calculating C_k in terms of the soliton parameters. We will see also that this integral is responsible

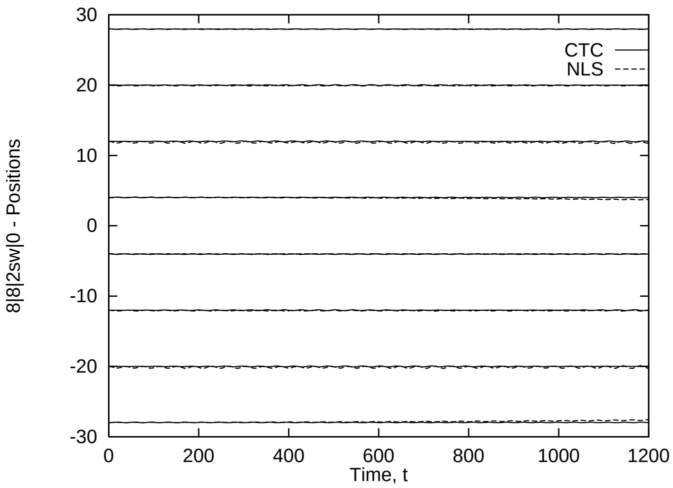

Fig. 1. Quasi-equidistant propagation of 8 solitons. The initial solitons parameters are $2\nu_1 = 2\nu_5 = 0.85$, $2\nu_2 = 2\nu_6 = 1.0$, $2\nu_3 = 2\nu_7 = 1.15$, $2\nu_4 = 2\nu_8 = 1.3$, $\xi_{k+1,0} - \xi_{k,0} = 8$, $\delta_{k,0} = 0.0$ and $\mu_{k,0} = 0$ for all $k = 1, \ldots, 8$.

for the 'secular' terms in C_k. As we will see below this integral is of order $\epsilon \ln \epsilon$ which allows for some additional simplifications. First we note that $z_k - z_n = 2(\nu_k - \nu_n)x + \alpha_{kn}$ and $\tilde{\phi}_k - \tilde{\phi}_n = 2(\mu_k - \mu_n)x + \phi_k - \phi_n$ where

$$\alpha_{kn} = 2\nu_n \xi_n - 2\nu_k \xi_k \simeq 2\nu_0(\xi_n - \xi_k),$$

and $\phi_k = \tilde{\phi}_k|_{x=0}$. Since $\nu_k - \nu_n$ and $\mu_k - \mu_n$ are of the order $\epsilon^{1/2}$ then we can neglect these differences in the integrand with the result:

$$\begin{aligned} J_{kn} &= \mathrm{e}^{\mathrm{i}(\phi_k - \phi_n)} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\cosh z_k \cosh(z_k + \alpha_{kn})} \\ &= \frac{\mathrm{e}^{\mathrm{i}(\phi_k - \phi_n)}}{\nu_k A_{kn}} \int_0^{\infty} \frac{\mathrm{d}y}{(y+1)(y + 1/A_{kn}^2)} \\ &= \mathrm{e}^{\mathrm{i}(\phi_k - \phi_n)} \frac{2A_{kn} \ln A_{kn}}{\nu_k (A_{kn}^2 - 1)}, \end{aligned} \tag{46}$$

where $A_{kn} = \exp(\alpha_{kn})$ and we have applied the change of variables $y = \mathrm{e}^{2z_k}$ and used the fact that $\mathrm{d}z_k/\mathrm{d}x = 2\nu_k$. With this simple change after the approximations described above all the integrals in C_k easily reduce to integrals of rational functions that are easy to calculate. In order to get the final result we have to keep in mind that $A_{k,k\pm1} \simeq \exp(2\nu_0(\xi_k - \xi_{k\pm1})) \simeq \epsilon^{\pm 1}$, which follow from the assumption (5).

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